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Temperature Extrapolation Mechanism for Two-Dimensional Heat Flow

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Unlike the investigations reported in the literature which are restricted to the simpler one-dimensional heat flow situation, the current study presents an analytical solution to the inverse problem that is applicable to two-dimensional conduction systems for geometries of arbitrary shape; heretofore intractable even for the simplest geometries. From theoretical considerations, temperatures can be predicted at discrete locations throughout the conducting medium, when input data such as thermocouple responses are known at several interior locations. Particularly, the transient temperature behavior may be readily established on any of the bounding surfaces by suitable interior thermocouple positioning. To facilitate computation of the desired temperatures, the theory allows for a temporal power series approximation of the input or thermocouple data, as is the customary practice in an experimental program. From transform techniques, the resultant theoretical expression for the prediction temperature is generated, and it appears as a summation of repeated error integrals. The form of the solution is convenient for numerical evaluation. Several numerical examples are presented as an indication of the accuracy of the theoretical results.

Nomenclature

$A_{l,m}{}^{ij}, C_l{}^{ij}, b_n{}^{ij}, c_{l,n}{}^{ij}$ = general coefficients
a_l, a_l' = free parameters, $a_l^2+(a_l')^2 = 1$
$f(t), g(t)$ = general functions of time
$f(\Delta), g(\delta)$ = defined by Eq. (5)
$i^n \operatorname{erfc} x$ = repeated error integral of variable x
l, m, n, q = summation indices
N = number of matched points per face
p = $(s/\alpha)^{1/2}$
s = Laplace transform variable
t = time variable
T = temperature
$\bar{T}$ = Laplace transform of temperature
x, y = space variables
α = thermal diffusivity
Δ = distance between thermocouples, $x_2 - x_1$
δ = distance between thermocouples, $y_2 - y_1$

Subscripts

1 = property along sampling path x_1 or y_1
2 = property along sampling path x_2 or y_2
i = property along sampling path x_i
j = property along sampling path y_j

Superscripts

i, j = property at position, $x = x_i$, $y = y_j$

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Introduction

FOR one-dimensional conduction systems, an analytical solution to the linear inverse problem is presented in Refs. 1 and 2. In the main, the temperature extrapolation procedure owes its success to the following features: first, the method does not rely upon numerical evaluation of complex integrals that ensue as a consequence of the application of transform techniques. Second, successive differentiation of interior temperature or heat flux data is not required; consequently the degradation associated with the values of the higher order time derivatives is eliminated. Last, internal temperature responses at two positions within the solid forms the basis for temperature extrapolation rather than heat flux and temperature data at one position. In any experimental program, a determination of the interior heat flux may prove to be impossible, whereas the thermocouple responses are readily available.

In Ref. 1, the literature pertaining to the one-dimensional inverse problem is reviewed in detail; however, a survey of the literature revealed that no information was available pertaining to extrapolation procedures for two- or three-dimensional systems. In the present investigation, a prediction method is presented for the first time, on two-dimensional conduction systems. The principles leading to this solution can be applied, as well, to the three-dimensional situation. In either case the transient temperature behavior may be determined at select locations throughout the medium, and on the bounding surfaces. Since the theoretical development that follows does not depend upon the geometry of the boundary, the extrapolation procedure may be considered as general, and therefore applicable for solids of arbitrary shape.

Before proceeding to the analysis, it should be noted that a successful extrapolation procedure requires that the temperatures must be known, a priori, throughout a closed region within the solid. The input temperatures or thermocouples are, therefore, positioned on the boundary of the contained region. Consequently, for the one-dimensional treatment, this requirement is easily satisfied since the two embedded thermocouples also represent the temperature of planar surfaces. Thus, the temperature behavior in the region between the thermocouples (or planes) is established. This is now a far more difficult matter for the two- (or three-) dimensional case, since thermocouple data must be provided at every point on the enclosing contour. In any experimental program, the data required would be formidable; consequently in the analysis that follows an extrapolation method is developed based upon thermocouples discretely positioned around the contour. Thus, as the number of temperature inputs increase, the inaccuracy of the method decreases. The method, therefore, establishes a mechanism for temperature prediction with sufficient accuracy based upon a limited number of thermocouple data.

Analysis of the Two-Dimensional Inverse Conduction Problem

For a solid with constant physical properties, the prediction temperature is determined from the differential equation

$$\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 = 1/\alpha\, \partial T/\partial t \tag{1}$$

with prescribed interior conditions. Since the thermocouples are positioned on the surfaces of an interior rectangle whose sides are $\Delta = x_2 - x_1$ and $\delta = y_2 - y_1$, the inputs are

$$T(x_i, y_j, t) = \sum_{n=1}^{n} b_n^{ij} t^n, \qquad i = 1, 2, ; j = 1, 2 \tag{2}$$

and

$$\begin{aligned} T(x, y_j, t) &= f(t), \quad x_1 < x < x_2 \\ T(x_i, y, t) &= g(t), \quad y_1 < y < y_2 \end{aligned} \tag{3}$$

Equation (2) represents the thermocouple data, as approximated by a power series, from the temperature sensors placed at the corners of the thermocouple rectangle, and Eq. (3), those units on the faces. The prediction or extrapolation process, therefore, applies to the region exterior to this rectangle.

By Laplace transform techniques, a solution to the differential system, the transform of Eqs. (1) and (2), is readily obtained. Accordingly, with the assumption of uniform initial conditions, the transform of the reduced temperature is

$$\begin{aligned} \bar{T}(x, y, p) = {} & \sum_{l=1}^{N+1} \frac{C_l^{11}(s)}{f(\Delta)g(\delta)} \left\{ e^{pa_l(x_1 - x)} - e^{-pa_l(x_1 - x + 2\Delta)} \right\} \times \\ & \left\{ e^{pa_l'(y_1 - y)} - e^{-pa_l'(y_1 - y + 2\delta)} \right\} - \sum_{l=1}^{N+1} \frac{C_l^{12}(s)}{f(\Delta)g(\delta)} \times \\ & \left\{ e^{pa_l(x_1 - x)} - e^{-pa_l(x_1 - x + 2\Delta)} \right\} \left\{ e^{pa_l'(y_1 - y - \delta)} - e^{-pa_l'(y_1 - y + \delta)} \right\} - \\ & \sum_{l=1}^{N+1} \frac{C_l^{21}(s)}{f(\Delta)g(\delta)} \left\{ e^{pa_l(x_1 - x - \Delta)} - e^{-pa_l(x_1 - x + \Delta)} \right\} \times \\ & \left\{ e^{pa_l'(y_1 - y)} - e^{-pa_l'(y_1 - y + 2\delta)} \right\} + \sum_{l=1}^{N+1} \frac{C_l^{22}(s)}{f(\Delta)g(\delta)} \times \\ & \left\{ e^{pa_l(x_1 - x - \Delta)} - e^{-pa_l(x_1 - x + \Delta)} \right\} \left\{ {pa_l'(y_1 - y - \delta)} - e^{-pa_l'(y_1 - y + \delta)} \right\} \end{aligned} \tag{4}$$

where

$$f(\Delta) = 1 - e^{-2pa_l\Delta} \quad \text{and} \quad g(\delta) = 1 - e^{-2pa_l'\delta} \tag{5}$$

In the preceding equation, the general term, $C_l^{ij}(s)$, represents a constant of integration to be determined, and the parameters, a_l and a_l', obey the relationship, $a_l^2 + (a_l')^2 = 1$. Further remarks pertaining to the selection of the free parameter, a_l, appears at the end of the paper. Since Eq. (4) must satisfy Eq. (2), it follows that

$$\sum_{l=1}^{N+1} C_l^{ij}(s) = \sum_{n=1}^{n} \frac{n!\, b_n^{ij}}{s^{n+1}} \tag{6}$$

To facilitate the prediction procedure, it is assumed that the individual term, $C_l^{ij}(s)$, may also be represented as

$$C_l^{ij}(s) = \sum_{n=1}^{n} \frac{n!\, c_{l,n}^{ij}}{s^{n+1}} \tag{7}$$

The rationale for this substitution is as follows: since Eq. (6) constitutes a temporal power series, Eq. (7) simply states that this series may be constructed from a sum involving individual series of the same form but with different coefficients, i.e.,

$$\sum_{n=1}^{n} b_n^{ij} t^n = \sum_{l=1}^{N+1} \sum_{n=1}^{n} c_{l,n}^{ij} t^n$$

The remaining condition, Eq. (3), is satisfied by point matching the inverse transform of Eq. (4) with individual temperature responses on the thermocouple rectangle. The quantity, N, that appears in the summation represents the numbers of matched points per face. If, for example, $N = 1$, then there is a set of four thermocouple data, one for each face, that Eq. (4) must satisfy. This, in turn, produces a set of equations involving the constants, $c_{l,n}^{ij}$, where the particular number of constants is determined by the degree, n, of the approximation polynomial. It should be noted that the corner located thermocouples correspond to a value of $N = 0$. In this manner, any number of thermocouple responses, appearing as a set of four, can be included in Eq. (4). Hence, the resultant expression for the temperature represents the transient temperature behavior within as well as on the rectangular region.

For temperature prediction purposes, it is now necessary to invert Eq. (4) for spatial coordinates that lie exterior to the thermocouple rectangle. Following the procedure shown in Ref. 1, enabling functions are selected so as to suppress the contributions of the position arguments in the exponential functions. In contrast to the one-dimensional inverse problem, identification of these functions is, momentarily, obscured by the two-dimensional character of the solution. This difficulty is resolved by capitalizing upon the separable nature of Eq. (4). Since the identification procedure, and hence the method of solution varies slightly for backward and forward extrapolation, the analysis will be divided accordingly.

For backward extrapolation, it is desired to predict the temperature anywhere within the regions, $(x < x_1, y < y_2)$, and $(x < x_2, y < y_1)$. Clearly, the prediction process can now proceed in any direction as contrasted to the one-dimensional situation where there is only one choice for the extrapolation path. In what follows, the prediction paths will be selected so that they are coincident with the sides of the thermocouple rectangle. The ensuing advantages are: identification of the required enabling functions, and simplification of the prediction process. Thus, temperature extrapolation along these select paths will be referred to as temperature sampling. As an illustration of this analysis procedure, the temperature will be determined along the sampling path, $y = y_j$. Equation (4), therefore, contracts to

$$\begin{aligned} \bar{T}(x \leqq x_2, y_j, p) = {} & \sum_{l=1}^{N+1} \sum_{n=1}^{n} \sum_{q=0,2,4}^{\infty} n!\, \frac{c_{l,n}^{1j}}{s^{n+1}} \times \\ & \left\{ e^{-pa_l[x - x_1 + q\Delta]} - e^{-pa_l[x_1 - x + (q+2)\Delta]} \right\} - \\ & \sum_{l=1}^{N+1} \sum_{n=1}^{n} \sum_{q=1,3,5}^{\infty} n!\, \frac{c_{l,n}^{2j}}{s^{n+1}} \times \\ & \left\{ e^{-pa_l[x - x_1 + q\Delta]} - e^{-pa_l[x_1 - x + q\Delta]} \right\} \end{aligned} \tag{8}$$

Consequently, the relationship between the temperature responses can now be identified as

$$\sum_{n=1}^{n} n!\frac{C_{l,n}^{1j}}{s^{n+1}} = \sum_{n=1}^{n} n!\frac{C_{l,n}^{2j}}{s^{n+1}} \sum_{m=1}^{m=n} A_{l,m}^{1j} e^{-ma_l p\Delta} \tag{9}$$

Substitution of Eq. (9) into Eq. (8) produces an invertible expression. Because the inversion method is now identical with that shown in Ref. 1, the details of the inversion procedure are omitted and only the final result will be indicated. Thus, the prediction temperature is for $x \leqq x_2$, $y = y_j$

$$T(x, y_j, t) = \sum_{l=1}^{N+1} \sum_{m=1}^{m=n} A_{l,m}^{1j} \sum_{n=1}^{n} n!\, c_{l,n}^{2j} \sum_{q=0,2,4}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{a_l\{x-x_1+(m+q)\Delta\}}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{a_l\{x_1-x+(m+q+2)\Delta\}}{2(\alpha t)^{1/2}}\right] - \sum_{l=1}^{N+1} \sum_{n=1}^{n} n!\, c_{l,n}^{2j} \sum_{q=1,3,5}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{a_l\{x-x_1+q\Delta\}}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{a_l\{x_1-x+q\Delta\}}{2(\alpha t)^{1/2}}\right], \quad \Delta \geqq x_1 \tag{10}$$

with

$$\sum_{n=1}^{n} c_{l,n}^{1j} t^n = \sum_{m=1}^{m=n} A_{l,m}^{1j} \sum_{n=1}^{n} n!\, c_{l,n}^{2j} (4t)^n i^{2n} \operatorname{erfc}\frac{a_l m\Delta}{2(\alpha t)^{1/2}} \tag{11}$$

In a similar fashion, the temperature may be sampled along the paths, $x = x_i$. To obtain the appropriate expressions for these directions, all that is required is that the order of the superscript indices are first interchanged, and then the index, j, and the quantity, Δ, replaced by the terms, i, and δ, respectively. Obviously, the spatial terms, y and y_1, are inserted for their counterparts, x and x_1, as well as the parameter a_l' for a_l. Accordingly, the resultant expression predicts the temperatures on the sampling paths, $x = x_i$, $y \leqq y_2$, when $\delta \geqq y_1$. For brevity, these expressions will not be generated since it is a simple matter to obtain them from the preceding equations.

For forward extrapolation, the temperature is predicted in the region exterior to the rectangle whose sides are, x_2, and y_2. Particularly, the temperature will be sampled along the paths, x_i, and, y_j, $i, j = 1, 2$. Fortunately, the generalized enabling functions associated with this procedure may be obtained from Eq. (9) after a cyclic permutation in the superscript indexed terms is performed. For example, those terms identified by the index 1, in the superscript should be replaced by the index 2. To complete the cycle, the coefficients containing the index 2 are replaced with the index 1. The indices identified by the letters, i or j, are left unchanged. Since Eq. (11) is the inverse transform of the enabling function, the preceding re-indexing criteria also applies to this equation. Upon substitution of the amended enabling functions into Eq. (4), the resultant expression can be inverted when, $x_2 \leqq x \leqq 2\Delta + x_1$, and $y_2 \leqq y < 2\delta + y_1$. Consequently, the generalized prediction temperature for forward extrapolation, along the sampling path $y = y_j$, is

$$T(x, y_j, t) = \sum_{l=1}^{N+1} \sum_{n=1}^{n} n!\, c_{l,n}^{1j} \sum_{q=0,2,4}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{a_l\{x-x_1+q\Delta\}}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{a_l\{x_1-x+(q+2)\Delta\}}{2(\alpha t)^{1/2}}\right] - \sum_{l=1}^{N+1} \sum_{m=1}^{m=n} A_{l,m}^{2j} \sum_{n=1}^{n} n!\, c_{l,n}^{1j} \sum_{q=1,3,5}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{a_l\{x-x_1+(m+q)\Delta\}}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{a_l\{x_1-x+(m+q)\Delta\}}{2(\alpha t)^{1/2}}\right], \quad \Delta > x_1 \tag{12}$$

with

$$\sum_{n=1}^{n} c_{l,n}^{2j} t^n = \sum_{m=1}^{m=n} A_{l,m}^{2j} \sum_{n=1}^{n} n!\, c_{l,n}^{1j} (4t)^n i^{2n} \operatorname{erfc}\frac{a_l m\Delta}{2(\alpha t)^{1/2}} \tag{13}$$

In accordance with the instructions that immediately follow Eq. (11), the expression for the prediction temperature along the sampling path, $x = x_i$, is obtained directly from the preceding equations.

From the theoretical considerations, thus far presented, it is apparent that the temperature may be predicted beyond the thermocouple locations, in either direction, when the desired location is on a sampling path. Particularly, the surface temperature behavior can be established at two positions on a boundary, and for industrial purposes this is customarily sufficient. Temperature extrapolation is also feasible for locations that do not lie on the sampling paths. Briefly, two possible methods will be discussed. As the first possibility, Eq. (4) may be inverted when the enabling functions are selected, as per Eqs. (11) and (13). This course of analysis has several distinct disadvantages. Since cancellation of terms appearing in the denominator does not occur, the final expression entails an additional summation. Similarly, none of the terms appearing in the braces vanishes; consequently the length of the expression cannot be reduced. The end result is, therefore, an unwieldy expression due to the presence of these terms. In addition, there is a theoretical limitation associated with backward extrapolation that should be noted. If the substitution of the enabling functions, Eqs. (11) and (13) is strictly adhered to, there exists two possibilities for the replacement of the term, $C_l^{11}(s)$. Regardless of which choice is made, the inversion procedure is not valid for a small region within the vicinity of the origin. When Eq. (11) is substituted for the value of $C_l^{11}(s)$ this region is beneath the line, $x + y = x_1 + y_1 - \Delta$. On the other hand, the area associated with Eq. (13) is bounded by the line, $x + y = x_1 + y_1 - \delta$. The inversion procedure does, however, apply for all points that are located on either line. Temperature prediction in the vicinity of the origin can be achieved by redefining the expression for the enabling functions. Replacement of the exponential argument by the quantity, $-a_l mp(\Delta+\delta)/(2)^{1/2}$ removes the undesirable restriction; however previous comments as to length still apply.

Broadening the temperature prediction process may also be accomplished via the temperature sampling method. Since the temperatures are available along the sampling paths y_1 and y_2, the prediction temperatures can be computed at the general positions, (x_o, y_1), and, (x_o, y_2), where, $0 < x_o < x_1$. With the generation of these new data points, temperature sampling in the y direction is now feasible. For this purpose, Eq. (10) must be amended so as to include the temperature responses at the new locations. Fortunately, this is readily effected by the replacement of the term, x_1, by the new sampling path, x_o, and computation of the terms $A_{l,m}^{01}$ and b_n^{0j} in accordance with Eqs. (2) and (9) where $i = 0$. Similarly, the temperature sampling procedure can be extended to sampling paths in the x direction. In principle, it is theoretically possible to generate the temperature responses at discrete locations throughout the entire two-dimensional field. This method contains none of the limitations associated with the first approach; however the coefficients A_m^{ij} and b_n^{ij} must be re-evaluated for each sampling path.

Numerical Examples

The principles of the inversion procedure, as derived in the preceding section, were applied to two geometrical configurations: a semi-infinite, right-angle corner, and a rectangular solid. The first geometry was selected because it provided among other considerations a rapid test of the method. For the corner, whose surface temperatures are maintained at unity, the theoretical expression for the temperature is the product of two error functions; consequently the theoretical solution is readily evaluated, and the resultant thermal symmetry reduces the computation time. As a practical example, the finite dimensional solid, the rectangle with sides, (1 ft, π ft), was chosen as the second illustration. Here, all the faces but one were considered to be maintained at zero temperature. The temperature on the remaining face, $x = 0$, increased linearly with time. Evaluation of the internal temperatures is lengthier than in the preceding case because of the slow convergence of the infinite sine series. In both instances, the thermocouples are positioned on the perimeter of an internal square whose sides are parallel to the

Table 1 Surface prediction temperatures for corner, $N = 0$

Time hr	$T(0,0.3)$ $n=5$	$T(0,0.3)$ $n=9$	$T(0,0.7)$ $n=5$	$T(0,0.7)$ $n=9$	$T(0.3,0)$ $n=5$	$T(0.3,0)$ $n=9$	$T(0.7,0)$ $n=5$	$T(0.7,0)$ $n=9$
0.10	1.0691	1.0159	1.0085	1.0201	1.0691	1.0168	0.8463	1.0201
0.20		1.0001		0.9907		1.0000		0.9907
0.30	0.9827	1.0003	1.0069	0.9996	0.9827	1.0004	1.0189	0.9996
0.40		0.9998		1.0000		0.9998		1.0000
0.50	1.0121	1.0005	0.9664	0.9970	1.0121	1.0005	0.9903	0.9970
0.60		0.9998		1.0022		0.9999		1.0022
0.70	0.9817	0.9997	1.0494	0.9996	0.9817	0.0007	1.0108	0.9992
0.80		1.0000		1.0005		1.0000		1.0005
0.90	1.0746	1.0070	0.9361	0.9764	1.0746	1.0072	0.9627	0.9764

cartesian axes. The distances between the corner-located thermocouples are selected as, $\Delta = \delta = 0.4$ ft, where the nearest corner of the thermocouple rectangle is at the point (0.3 ft, 0.3 ft). Accordingly, the positioning of the thermocouple square satisfies the requirement, $\Delta > x_1$, and $\delta > y_1$.

In any experimental program, the temperature traces would be obtained directly from the aforementioned thermocouple responses. However in this instance, the temperature behavior at the thermocouple locations are generated from the analytical solution to the appropriate direct conduction problem. In turn, these transient temperature responses at discrete locations on the square's perimeter are approximated by a nth degree polynomial for a time range which is divided into equal increments. Thus the numerical values for the constants, b_n^{ij}, in Eq. (2) are obtained, as well as the functions $f(t)$ and $g(t)$ in Eq. (3).

For the first approximation, the temperature responses at the corners are only utilized; hence the number of matched points per face excluding the corners is zero, i.e., $N = 0$. The free parameter, a_l, is determined from the relationship, $a_l = (1+l)^{-1/2}$, i.e., $a_1 = (2)^{-1/2}$, and the inverse of Eq. (8) computed. For brevity's sake, this inverse is not shown explicitly since it is a simple matter to invert the equation as it appears. Alternatively, the explicit formulation may be obtained directly from Eq. (10) in the following manner: in the first summation replace, $A_{l,m}^{1j}$ by unity, $c_{l,n}^{2j}$ by $c_{l,n}^{1j}$ and in the arguments for the repeated error integral the quantity, m, by zero. These changes are all that are required, and the final expression represents the inverse of Eq. (8). Similar substitutions can be made in the amended version of Eq. (8) for the sampling path, $x = x_i$.

Returning to the computation scheme, the calculated temperatures for $N = 0$, are compared with the thermocouple responses at the appropriate positions on the perimeter. The magnitude of the deviations will dictate whether further approximations are warranted. Should this be the case, then the inverse of Eq. (8) is point-matched at discrete perimeter locations, where a set of four points, one for each face, corresponds to $N = 1$. In this manner a system of simultaneous equations is generated and the values of the coefficients, $c_{l,n}^{ij}$ are determined. The temperature deviations are computed again, and if necessary the point-matching method is extended to embrace additional thermocouple locations. Once sufficient accuracy is established, the temperature may be extrapolated along the sampling paths in accordance with Eqs. (10) and (12).

Table 2 Surface prediction temperatures for rectangle, $N = 0$

Time hr	$T(0,0.3)$ $n=4$	$T(0,0.7)$ $n=4$	$T(0.3,0)$ $n=4$	$T(0.7,0)$ $n=4$
0.10	0.0630	0.0798	0.0392	0.0062
0.15	0.0939	0.1210	0.0558	0.0098
0.20	0.1275	0.1664	0.0719	0.0132
0.25	0.1634	0.2151	0.0877	0.0166
0.30	0.1947	0.2572	0.1038	0.0195

Table 3 Surface prediction temperatures for rectangle, $N = 1$

Time hr	$T(0,0.3)$ $n=4$	$T(0,0.7)$ $n=4$	$T(0.3,0)$ $n=4$	$T(0.7,0)$ $n=4$
0.10	0.0803	0.1345	0.0158	−0.0041
0.15	0.1379	0.1478	0.0092	−0.0041
0.20	0.1946	0.1949	0.0041	0.0003
0.25	0.2453	0.2560	0.0011	0.0139
0.30	0.2986	0.2911	−0.0002	0.0079

The results of the computations for the two cases are shown in Tables 1–4, where the thermal diffusivity, α, was selected as, 1.25 ft^2/hr. Due to the placement of the thermocouple rectangle, the surface prediction temperatures obtained were in close proximity to the origin. As shown in Table 1, the extrapolated temperatures compare favorably with the theoretical value of unity for each face. It is apparent that in this instance as the power of the approximation polynomial, $n = m$, increases the accuracy of the method improves. It should be noted that these results were obtained from a first approximation. In other words, the temperature responses at the corners were only utilized, since the inverse of Eq. (8) deviated less than 2% in value from the traces associated with the discretely located thermocouples. In the main, the deviations were considerably less than the 2%. Accordingly, further correction was deemed unnecessary; hence additional point-matches were not required and the temperatures were extrapolated based upon the corner-located thermocouples.

Tables 2 and 3 represent the computation for a rectangular solid with sides (1 ft, π ft). Three of the rectangle's faces were maintained at zero temperature, and the remaining face, $x = 0$, varied linearly with time, i.e., $T(o, y, t) = t$. As shown in Table 2, temperature extrapolations based upon the corner thermocouples was not successful. This was anticipated, since the results from Eq. (8) revealed temperature deviations of about 30% at the nonmatched points on the thermocouple perimeter. Consequently, the temperatures were point-matched at an additional position, i.e., $N = 1$, and the inverse Eq. (8) recomputed. The resultant deviations were less than 9% and in the main in the vicinity of 2%. The most serious deviation occurred for the first time value; $t = 0.10$ hr, at two of the eight positions that were used as nonmatched points. Comparison of Table 3 with Table 2, clearly indicates the improvement in the accuracy of the extrapolated results for a one-point match. Similar results were obtained for forward extrapolation, Eq. (12), and for brevity's sake these results are not included at this time.

Conclusion

An analytical solution is presented for the inverse conduction problem which incorporates thermocouple data from interior thermocouple positions. For two-dimensional geometries, the temperature can be predicted throughout the solid by the mechanism of temperature sampling. It is interesting to note that along any sampling path, the extrapolation procedure resembles the one-dimensional method developed in Ref. 1. Equation (10) is the one-dimensional solution when the quantity, a_l, is replaced by unity, and only corner-located thermocouples are utilized, i.e., $N = 0$. For other values of N, which correspond to additional matched points, the constants, $c_{l,n}^{ij}$, are generated from a set of simultaneous equations. Accordingly, as the number of points increase, the numerical computations are greatly enlarged. This occurs because the method requires a double approximation, one for the spatial variation along the thermocouple perimeter, and the other for the temporal series replacement at each point. For widely spread thermocouples, which produces a large thermocouple rectangle with many points to be matched, considerable computer time would be necessary for surveying the temperatures along the distant sampling paths. It is more expeditious in this instance to position thermocouples so that

Table 4 Deviation of internal predicted temperatures for rectangle, $N = 1$

Time hr	(0.15, 0.3) $n = 4$	(0.15, 0.7) $n = 4$	(0.3, 0.15) $n = 4$	(0.7, 0.15) $n = 4$
0.10	0.572×10^{-2}	-1.288×10^{-2}	-0.576×10^{-2}	0.059×10^{-2}
0.15	0.406×10^{-2}	-0.105×10^{-2}	-0.392×10^{-2}	0.011×10^{-2}
0.20	0.143×10^{-2}	0.460×10^{-2}	-0.257×10^{-2}	0.158×10^{-2}
0.25	0.283×10^{-2}	-0.430×10^{-2}	-0.210×10^{-2}	-0.568×10^{-2}
0.30	0.108×10^{-2}	1.410×10^{-2}	-0.176×10^{-2}	-0.167×10^{-2}

several smaller thermocouple rectangles ensue and less points to be matched. A smaller thermocouple rectangle suggests that the temperatures are sampled in a confined region, i.e., near a corner. In industrial applications, a rapid estimate of the temperatures for a surface zone is often required. The sampling technique thus described is particularly suited to meet these needs. Further reduction in computer time may be achieved by selection of a smaller value for the degree in the temporal power series. The tabulated values in Table 3 demonstrate that a fourth-degree polynomial approximation fitted by the least squares techniques for five temperature values produced sufficient accuracy; consequently the computations were terminated.

The extrapolated temperature values were also computed at the intermediate positions, $x = 0.15$, or $y = 0.15$, along the appropriate sampling path. The deviations from the theoretical results are shown in Table 4. It should be noted that the deviation, or error, increases as the prediction process is employed further from the thermocouple rectangle. This can be substantiated when deviations related to Table 3 are compared to Table 4. Accordingly, it is recommended that the thermocouple rectangle be positioned in close proximity to the region of interest.

As previously noted, when the free parameters, a_l and a_l', are selected so that they satisfy the relationship, $(a_l)^2 + (a_l')^2 = 1$, then Eq. (4) is a solution to the differential equation. At first glance, it would appear this constitutes all the information available for the selection of a_l or a_l'. However, it should be pointed out that only real values for a_l or a_l' are allowable, since any other choice would result in periodic functions. A function possessing periodicity cannot be extrapolated beyond the range of the thermocouple locations; consequently it does not lend itself to the temperature prediction process. Inspection of Eq. (4) also reveals that negative values for a_l or a_l' yield an expression that is identical to that for positive values. Accordingly, the suitable selection for the parameters is in the range $0 \leqq a_l$ or $a_l' \leqq 1$. For the first approximation, the parameters are selected so that the temperature predictions are weighted equally in both directions, i.e., $a_1 = a_1' = (2)^{-1/2}$. All other values for a_l are arbitrarily selected as $a_l = (1 + l)^{-1/2}$, since this relationship satisfies the preceding requirements.

Experience gained from continued application of the method indicates that the method is more sensitive than the extrapolation procedure for the one-dimensional situation. This, in a large measure, is due to the double approximations previously referred to. Hence all computations were performed in a double precision mode. It should be noted that the success of the method is keyed directly to the temporal series replacement represented by Eq. (13). For this replacement, it is obvious that adequate thermocouple traces must be available at the two positions involved, $(2, j)$ and $(1, j)$. If, for example, the thermocouple rectangle were positioned in a region where some or all the thermocouples did not respond, then, any attempt to employ Eq. (13) would result in extremely large (or small) values of A^{2j}. The credibility of these computed values would be suspect and the extrapolation procedure would not apply. It is envisaged that there may be thermal situations, such as those exhibiting rapid transients or large thermal lags, for which the thermal sensors on the thermocouple rectangle do not respond. In these instances, it is necessary to reposition the location of the rectangle, so that adequate thermocouple traces can be obtained, and in turn suitable series replacement are represented by Eq. (13).

In conclusion, an analytical solution has been obtained for the inverse conduction problem which utilizes thermocouple data from interior positions. The method derives its success from the feature of point-matching temperature responses along the perimeter of the thermocouple rectangle. Since there are no restrictions upon the boundary surface, the method is also applicable to irregular shaped boundaries. By inspection, a method of temperature extrapolation for three-dimensional geometries is now feasible. Here, the thermocouples are located along the corners of an interior rectangle parallelpiped, and the sampling paths are chosen so that they coincide with the edges of this geometrical shape. Since the data must be available for many more points than in the two-dimensional situation, it can be anticipated the computation process would be lengthy. In addition, suitable adjustments would have to be made in Eq. (10) to accommodate the third space variable.

References

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